ON A CONJECTURE OF LINDENSTRAUSS*

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ABSTRACT

It is proved that each n-dimensional centrally symmetric convex polyhedron admits a 2-dimensional central section having at least $2n$ vertices. Some other related results are obtained and some unsolved problems are mentioned.

The following conjecture of Joram Lindenstrauss was transmitted to me by Branko Grünbaum:

(Σ) *If P is an n-dimensional centrally symmetric convex polyhedron, then some 2-dimensional central section of P has at least $2n$ vertices.*

I present here a proof of this conjecture as well as a dual statement (I) about affine images of polyhedra. In fact, the simplest approach to (Σ) seems to be by way of (I), using the duality between sections and affine images as in [2,3].

Let us begin with the proof of (I), where of course (Σ) and (I) both require that $n \geqq 2$.

(I) *If P is an n-dimensional centrally symmetric convex polyhedron, then some 2-dimensional affine image of P has at least $2n$ vertices.*

Proof. The assertion is obvious for $n = 2$. Suppose it is known for $n = k$ and consider the case $n = k + 1$, where we assume without loss of generality that P lies in a $(k + 1)$-dimensional real vector space E and is centered at the origin 0 of E. We wish to show that some 2-dimensional linear image of P has at least $2k + 2$ vertices. Let V be the set of all vertices of P and let ξ be a linear transformation of E onto a k-dimensional vector space F such that ξ is biunique on V. Then ξP is a k-dimensional centrally symmetric convex polyhedron, so (by the inductive hypothesis) there exists a linear transformation η' of F onto a 2-dimensional vector space G such that the convex polygon $Q' = \eta'\xi P$ has at least $2k$ vertices.

Let $\mathscr{L}(F, G)$ (resp. $\mathscr{L}_0(F, G)$) denote the space of all linear transformations of F into G (resp. onto G); the spaces $\mathscr{L}(E, G)$ and $\mathscr{L}_0(E, G)$ are similarly defined. In the following paragraphs, it is convenient to topologize the spaces E, F, and G by means of Euclidean metrics, the spaces $\mathscr{L}(F, G)$ and $\mathscr{L}(E, G)$ by means of uniform norms, and the space $\mathscr{K}$ of all convex polygons in G by means of the

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Hausdorff metric. However, at the cost of some extra effort it would be possible to dispense with these assumptions and carry out the proof for vector spaces over an arbitrary ordered field.

Noting that the function $\psi\xi P \mid \psi \in \mathscr{L}(F,G)$ maps $\mathscr{L}(F,G)$ continuously into $\mathscr{K}$, that the function (number of vertices of K) $\mid K \in \mathscr{K}$ is lower semicontinuous on $\mathscr{K}$, and that the set $J = \{\psi \in L_0(F,G) : \psi$ is biunique on $\xi V\}$ is a dense open subset of $\mathscr{L}(F,G)$, we see the possibility of choosing η in J close to η' such that the convex polygon $Q = \eta\xi P$ has at least $2k$ vertices. When Q has at least $2k+2$ vertices, there is no problem. When Q has only $2k$ vertices, we shall produce in G another linear image Q^+ of P which has more vertices than Q.

Production of Q^+ depends on the following simple fact: (†)

If Q is a convex polygon in the plane G, A is the set of all vertices of Q, and B is a finite subset of $Q \sim A$, then there exists $\varepsilon > 0$ such that whenever B^+ is a finite set for which $B^+ \not\subset Q$ but B^+ lies in the ε-neighborhood of B, then the polygon $Q^+ = \operatorname{conv}(A \cup B^+)$ has more vertices than Q has. Here it is essential that the set B should not include any vertex of Q, and this accounts for our interest in transformations of E which are biunique on the set V of vertices of P.

Suppose Q has only the $2k$ vertices $\pm q_1, \ldots, \pm q_k$. For each i there is a unique vertex p_i of P such that $\eta\xi p_i = q_i$. Let H be a k-dimensional linear subspace of E such that $\{p_1, \ldots, p_k\} \subset H$ and let ζ denote the restriction of $\eta\xi$ to H. Then of course $\zeta(P \cap H) = Q$. Let $u \in P \sim H$, so that each point x of E admits a unique expression in the form $x = x' + x''u$ with $x' \in H$ and $x'' \in R$ (real numbers). For each point $z \in G$, let the transformation $\phi_z \in \mathscr{L}_0(E,G)$ be defined as follows:

$$\phi_z(x' + x''u) = \zeta x' + x''z.$$

Note that with $z_0 = \eta\xi u$, we have

$$\phi_{z_0}(x' + x''u) = \zeta x' + x''\eta\xi u = \eta\xi x' + x''\eta\xi u = \eta\xi(x' + x''u), \text{ so } \phi_{z_0} = \eta\xi.$$

Let G_0 denote the set of all $z \in G$ for which the transformation ϕ_z is biunique on V. Then $z_0 \in G_0$, and we claim further that the set $G \sim G_0$ is finite. Indeed, if $z \in G \sim G_0$ there exist $v \in V$ and $w \in V$ such that $v \neq w$ but $\phi_z v = \phi_z w$, whence $\zeta(v' - w') = (w'' - v'')z$. If $v'' = w''$, then $\zeta v' = \zeta w'$ and $\eta\xi v = \eta\xi w$, contradicting the choice of η. If $v'' \neq w''$, then $z = (w'' - v'')^{-1}\zeta(v' - w')$. Thus there are only finitely many possibilities for $z \in G \sim G_0$.

Let $z_1 \in G_0 \sim Q$. Since $z_0 \in G_0$ and the set $G \sim G_0$ is finite, there exists $z_{1/2} \in G_0$ such that the segments $[z_0, z_{1/2}]$ and $[z_{1/2}, z_1]$ both lie entirely in G_0. Define

$$z_\lambda = (1 - 2\lambda)z_0 + (2\lambda)z_{1/2} \qquad \text{for } 0 \leqq \lambda \leqq 1/2, \text{ and}$$

$$z_\lambda = (2 - 2\lambda)z_{1/2} + (2\lambda - 1)z_1 \qquad \text{for } 1/2 \leqq \lambda \leqq 1.$$

Then the function $\phi_{z_\lambda}P \mid \lambda \in [0,1]$ is a continuous mapping of $[0,1]$ into $\mathscr{K}$. Note that

$\phi_{z_\lambda}P \supset \zeta(H \cap P) = Q$ for all $\lambda \in [0,1]$, and that

$\phi_{z_0}P = \eta\xi P = Q$ while

$u \in P$ and $\phi_{z_1}u = z_1 \notin Q$.

Let μ be the least upper bound of those $\lambda \in [0,1]$ for which $\phi_{z_\lambda}P = Q$. Then $\mu < 1$ and $\phi_{z_\mu}P = Q$, but there are values of λ arbitrarily close to μ for which Q is properly contained in the polygon $\phi_{z_\lambda}P$. Of course $\phi_{z_\lambda}P = \operatorname{conv}\phi_{z_\lambda}V$ for all $\lambda \in [0,1]$, and since $z_\mu \in G_0$ we have

$$\phi_{z_\mu}(V \sim \{\pm p_1, \dots, \pm p_k\}) \subset Q \sim \{\pm q_1, \dots, \pm q_k\}.$$

Applying the italicized statement (†) above, we see the existence of λ close to μ for which the convex polygon $Q^+ = \phi_{z_\lambda}Q$ has more vertices than Q and hence, being centrally symmetric, has at least $2k + 2$ vertices. The proof is now completed by mathematical induction.▌

It seems worthwhile to provide a more metric form for (I).

(Π) *If P is an n-dimensional centrally symmetric convex polyhedron in E^n, there is a 2-dimensional plane T in E^n such that the orthogonal projection π of E^n onto T carries P onto a convex polygon πP having at least $2n$ vertices.*

Proof. By (I) there is a linear transformation τ of E^n onto E^2 such that τP has at least $2n$ vertices. Let $T^1 = \tau^{-1}(0)$ and let T be the orthogonal supplement of T^1 in E^n.▌

Proof of (Σ). Let P be an n-dimensional centrally symmetric convex polyhedron, centered at the origin of an n-dimensional real vector space E. Let $\langle\,,\rangle$ be an inner product on E and let P^0 be the polar body $\{y \in E : \langle x, y\rangle \leqq 1$ for all $x \in P\}$. Then of course P^0 is an n-dimensional centrally symmetric convex polyhedron, and by (Π) there exists a two-dimensional linear subspace T of E such that the convex polygon πP^0 has at least $2n$ vertices, where π is the orthogonal projection of E onto T. Since

$$T \cap (P \cap T)^0 = T \cap \operatorname{cl}\operatorname{conv}(P^0 \cup T^0) = T \cap (P^0 + T^0) =$$
$$T \cap (\pi P^0 + T^0) = \pi P^0,$$

it follows that the polygon $P \cap T$ has at least $2n$ vertices. (The relevant basic results on the polarity $\circ$ may be found in [1] and in [2].)▌

In the absence of symmetry assumptions, the methods used for (Π) and (Σ) lead to similar results in which the number $2n$ is replaced by $n + 1$. In the nonsymmetric version of (Σ), there exist 2-dimensional sections with at least $n + 1$ vertices through *each* interior point of the n-dimensional convex polyhedron P.

There remain many interesting problems concerning the numbers of faces of sections or projections of convex polyhedra. Some of these may be formulated as follows. Suppose $\mathscr{P}$ is an indexed family $\{(P_\iota, X_\iota): \iota \in I\}$, where, for each $\iota \in I$, P_ι is an n-dimensional convex polyhedron in E^n and X_ι is a nonempty subset of E^n. For $0 \leqq j < k \leqq n$, let $\Sigma(\mathscr{P}, k, j)$ denote the largest number r such that for all $\iota \in I$ and

(*) for all $x \in X_\iota$, there is a k-dimensional flat F through x for which the intersection $P_\iota \cap F$ is a k-dimensional convex polyhedron having at least r j-dimensional faces.

For various choices of $\mathscr{P}$, and for given $j < k$, it would be of interest to determine the number $\Sigma(\mathscr{P}, k, j)$ and to describe the "minimal members" of $\mathscr{P}$—that is, those members (P_ι, X_ι) of P such that (*) fails for $r > \Sigma(\mathscr{P}, k, j)$. Of special interest are the family $\mathscr{S}_n$ of all pairs $(P, \{x\})$ for which P is an n-dimensional convex polyhedron which is centrally symmetric about x, and the family $\mathscr{P}_n$ of all pairs $(P, \operatorname{int} P)$ where P is an n-dimensional convex polyhedron.

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